6D Bounded linear functionals

Linear functional: $\varphi \in B(V, F)$ a linear map from V to the field.

Note F is complete as a vector space ($\mathbb{R}$ or $\mathbb{C}$ with norm $|\cdot|$)
So
Continuity $\Leftrightarrow$ Boundedness!

Null space: $\text{null}(T) = \{x \in V : Tx = 0\}$

If T is continuous: $T^{-1}\{0\} = \text{null}(T)$ is closed.

☆ HW / Exam

How about the converse? False in general when $T: V \longrightarrow W$.
Recall $V = \{\vec{a} \in \ell^\infty : \exists\, \mathbb{N} \text{ st } a_n = 0 \ \forall\, n \geq N\}$ with $\|\ \|_\infty$ norm.
$T(\vec{a}) = (a_1, 2a_2, \ldots)$
$T(\vec{a}) = 0 \Leftrightarrow a_1 = 0,\ a_2 = 0, \ldots \Rightarrow \text{null}(T) = \{\vec{0}\}$. This is closed because it has only the constant sequence in it converging to $\vec{0}$.

→ However CONVERSE is true for linear funchals.

→ Remark: What properties do you require of W for this to be true?

Prob: Let $\varphi: V \longrightarrow F$ linear $\varphi \neq 0$. Then TFAE
(i) $\varphi \in B(V, F) =: V'$ (dual space)
(ii) φ contin
(iii) $\text{null}(\varphi)$ closed subspace of V
(iv) $\overline{\text{null}(\varphi)} \neq V$

(ii) $\Leftrightarrow$ (i) previously proved

(ii) $\Rightarrow$ (iii) obvious let $x_n \in null(\varphi)$ $x_n \to x$ $\Rightarrow \varphi(x) = 0$

(iii) $\Rightarrow$ (i) Suppose not (i), ie φ not bounded.

Then $\exists$ $\|x_n\| = 1$; $|\varphi(x_n)| \to \infty$

$$\frac{x_n}{\varphi(x_n)} \to 0 \qquad a_n = \underbrace{\frac{x_1}{\varphi(x_1)} - \frac{x_n}{\varphi(x_n)}}_{\in\, null(\varphi)} \qquad \varphi(a_n) = 0$$

$$a_n \to \frac{x_1}{\varphi(x_1)}$$

$\lim \varphi(a_n) \to \varphi\left(\frac{x_1}{\varphi(x_1)}\right) = 1$ $\Rightarrow$ null(φ) not closed

(iii) $\Rightarrow$ (iv) $\overline{null(\varphi)} \subset V$ but $\varphi \neq 0$ so $\overline{null(\varphi)} \neq V$.

not (iii) $\exists$ $x_n \in null(\varphi)$ $x_n \to x$ $\varphi(x) \neq 0$.

Take any $y \in V$ $\quad \underbrace{y - \frac{\varphi(y)}{\varphi(x)} x}_{z\, = \,\in\, null(\varphi)} + \frac{\varphi(y)}{\varphi(x)} x = y$

But $\quad \underbrace{\frac{\varphi(y)}{\varphi(x)} x_n}_{null(\varphi)} \to \frac{\varphi(y)}{\varphi(x)} x$

And $\quad \underbrace{z + \frac{\varphi(y)}{\varphi(x)} x_n}_{null(\varphi)} \to y \quad \Rightarrow y \in \overline{null(\varphi)}$

Linear independence and bases

A family $\{e_k\}_{k\in\Gamma}$ in V is $e : \Gamma \longrightarrow V$ a function.

Γ can be uncountable.

Families in V

→ Independent if $\nexists\ \underset{|\Omega|<\infty}{\Omega \subset \Gamma}$, $\Omega \neq \phi$ and a family

$\{\alpha_j\}_{j\in\Omega} \in F^\Gamma \setminus 0$ st $\sum_{j\in\Omega} \alpha_j e_j = 0$

→ (finite) span of Γ = $\{ \sum_{j\in\Omega} \alpha_j e_j : \Omega$ finite, $\Omega \subset \Gamma$, $\{\alpha_j\}_{j\in\Omega} \in F$

→ Finite dim V if $\exists\ \{e_j\}_{j\in\Gamma}$ such that span$(\{e_j\}_{j\in\Gamma}) = V$ and Γ finite.

→ $\{e_j\}_{j\in\Gamma}$ is a basis if it spans V and is independent.

Ex: Let $x \in V$ and $\{e_j\}_{j\in\Gamma}$ be a basis. Prove that $\exists!\ \Omega$, $\{\alpha_j\}_{j\in\Omega}$
$x = \sum \alpha_j e_j$

Remark: Sometimes we consider $\overline{\text{span}(\{e_j\}_{j\in\Gamma})}$ as in Hilbert spaces. If $\overline{\text{span}(\{e_j\}_{j\in\Gamma})} = V$ then we also called it a basis (conventionally). So to distinguish them from the "finite" bases we are considering we typically call them HAMEL bases.

Maximal elements: Let $\mathcal{A} \subset 2^V$, a collection of subsets.

Γ is a maximal element if $\nexists\ \Gamma' \in \mathcal{A}$ st $\Gamma \subsetneq \Gamma'$

(the $\subsetneq$ can be omitted)

Ex: Let $k\mathbb{Z} = \{km : m \in \mathbb{Z}\}$

Let $\mathcal{A} = \{k\mathbb{Z} : k = 2, 3, \ldots\}$

Claim: Γ is a maximal element iff $\Gamma = p\mathbb{Z}$ p prime.

Zorns lemma

$\mathcal{C} \subset 2^V$ is a chain if $a, b \in \mathcal{C} \Rightarrow$ either $a \subseteq b$ or $b \subseteq a$. (Totally ordered in inclusion)

Ex: $\mathcal{C} = \{4\mathbb{Z}, 6\mathbb{Z}\}$ is not a chain

$\mathcal{C} = \{2^n\mathbb{Z} : n \in \mathbb{Z}^+\}$ is a chain

Zorn's lemma: Let $\mathcal{A} \subset 2^V$ have the property that if $\mathcal{C} \in \mathcal{A}$ is a chain then the union of all elements in $\mathcal{C} \in \mathcal{A}$ as well. Then $\mathcal{A}$ has a maximal element.

Prop: Every vector space has a basis.

Pf: Let $\mathcal{A}$ = collection of all indep sets of vectors in V. Let $\mathcal{C}$ be a chain in $\mathcal{A}$. Let $B = \bigcup_{X \in \mathcal{C}} X$. Take any finite collection of vectors in B.

$= \{e_1, \dots, e_n\}$. Each $e_i \in X_i \in \mathcal{C}$. But since $\mathcal{C}$ is a chain we can prove $\exists X$ st $\{e_1, \dots, e_n\} \in X$ (by induction if its not obvious)

$\Rightarrow$ B is a linearly indep family. $\Rightarrow B \in \mathcal{A}$ by definition.

By Zorn, $\mathcal{A}$ has a maximal linearly indep. set; ie a basis.

Prop: Every ∞-dimensional space has a discontinuous linear functional.

Pf: V has a basis $\{e_j\}_{j\in\Gamma}$. Choose a countable subcollection

R as follows: (i) Γ is nonempty so let x_1 be an element of Γ.
(ii) $\Gamma \setminus \{x_1, \dots x_n\}$ is nonempty so let $x_{n+1} \in \Gamma \setminus \{x_1, \dots, x_n\}$

$R = \{x_i\}_{i=1}^{\infty}$

Remark: Are we using AoC to select x_i? Is more like "existential instantiation." $(\exists x \in \mathcal{C})$.

Let $\varphi(x_j) = j\|x_j\|$, and $\varphi\left(\sum_{j\in\Omega} \alpha_j e_j\right) = \sum_{j\in\Omega\cap\mathbb{Z}^+} \alpha_j j\|e_j\|$

Clearly φ is unbounded.

<u>Hahn-Banach theorem</u>

How about continuous functionals? Harder to show they exist.

For any $U \subset V$, a subspace. Define for any $h \in V$

Let $U + \mathbb{R}h = \{f + \alpha h : f \in U, \alpha \in \mathbb{R}\}$

<u>Lemma:</u> If $\psi : U \to \mathbb{R} \in U'$ and let $h \in V \setminus U$. Then

$\exists\, c \in \mathbb{R}$ st $\varphi : U + \mathbb{R}h \longrightarrow \mathbb{R}$ defined by

$$\varphi(u + \alpha h) = \psi(u) + \alpha c$$

satisfies $\|\varphi\| = \|\psi\|$

Pf: Clearly $\|\varphi\| \geq \|\psi\|$ since $\varphi|_U = \psi$

To show

$$|\varphi(x+\alpha h)| = |\psi(x)+\alpha c| \leq \|\psi\|\,\|x+\alpha h\| \qquad \forall x \in \mho,\ \alpha \in F$$

$$\Leftrightarrow \quad \left|\psi\left(\frac{x}{\alpha}\right) + c\right| \leq \|\psi\|\,\left\|\frac{x}{\alpha}+h\right\| \qquad \forall x \in \mho,\ \alpha \in F$$

$$\Leftrightarrow \quad |\psi(y)+c| \leq \|\psi\|\,\|y+h\| \qquad \forall y \in \mho$$

(using $\mho$ is a subspace

$$-\|\psi\|\,\|y+h\| \leq \psi(y)+c \leq \|\psi\|\,\|y+h\|$$

$$-\|\psi\|\,\|y+h\| - \psi(y) \leq c \leq \|\psi\|\,\|y+h\| - \psi(y) \qquad \forall y$$

We show

$$\sup_y \left[-\|\psi\|\,\|y+h\| - \psi(y)\right] \leq \inf_x \|\psi\|\,\|x+h\| - \psi(x)$$

Then such a c will exist.

$$\psi(x) - \psi(y) \leq \|\psi\|\left(\|x+h\| + \|y+h\|\right)$$

$$\psi(x) - \psi(y) = \psi(x-y) \leq \|\psi\|\,\|x-y\|$$

and the rest follows from Δ inequality!

Given $T: V \to W$

<u>Graph of T</u> : $\text{graph}(T) \subset V \times W = \{(v, Tv) : v \in V\}$

<u>Ex</u>

1) T is linear iff $\text{graph}(T)$ is a subspace

2) $U \subseteq V$ $S: U \to W$ Then T extends S iff $\text{graph}(S) \subseteq \text{graph}(T)$

3) $T: V \to W$ $c \in [0, \infty)$ then $\|T\| \leq c \iff \|x\| \leq c\|y\|$ $\forall (x,y) \in \text{graph}(T)$

<u>Hahn-Banach Theorem</u>

Let $(V, \| \ \|)$ normed, $U \subset V$ subspace, $\varphi: U \to F$ is a bounded functional. Then φ can be extended to V st $\|\varphi\| = \|\varphi|_U\|$

Pf: Recall extension lemma only held for $\mathbb{R}$. So first prove for $\mathbb{R}$.

$$A = \{E \subset V \times \mathbb{R} : E = \text{graph}(\varphi) \text{ on some subspace of } V \text{ for some } \varphi, \ \text{graph}(\varphi) \subseteq E, \ |\alpha| \leq \|\varphi\| \|x\| \ \forall (x, \alpha) \in E\}$$

"$\varphi(x)$"

In other words, A is the graph of all extensions of φ st $\|\tilde{\varphi}\| \leq \|\varphi\|$

Ex: Take $\mathcal{C} = \{E_1, E_2, \dots\}$ a chain in A.

Then consider $\cup E_i = E$. Take $\alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1 + \beta x_2, \varphi(\alpha x_1 + \beta x_2)) \in E$

$\Rightarrow$ E is a subspace.

let $\varphi(x) = \varphi_i(x)$ if $x \in E_i$

this is a linear functional defined on the union of the domains of E_i. Call it B. One can check that E is the graph of φ on B. Its clear this φ satisfies the condition as well.

So $\exists$ a maximal element of A, it is the graph of some φ, and φ must extend ψ with $\|\varphi\| = \|\psi\|$

By the extension lemma, the domain of φ must be all of V.

Now for $F = \mathbb{C}$. Define $\psi_1 : U \longrightarrow \mathbb{R}$

$$\psi_1(x) = \operatorname{Re} \psi(x) \qquad \forall x \in U$$

ψ_1 is a linear map on $(U, \mathbb{R})$ and

$$|\psi_1(x)| \leq |\psi(x)| \qquad \forall x \in U$$

complex

$$\Rightarrow \quad \|\psi_1\| \leq \|\psi\|$$

Note that ψ can be written entirely in terms of ψ_1

$$\begin{aligned} \psi(x) &= \operatorname{Re} \psi(x) + i \operatorname{Im} \psi(x) \\ &= \psi_1(x) + i \operatorname{Im}(-i\,\psi(ix)) = \psi_1(x) - i\,\psi_1(ix) \end{aligned}$$

Think of $(V, \mathbb{R})$ as a vector space: (i) Closure ✓
(ii) Inverses ✓ (iii) additive identity $(\vec{0} + v = v)$ same as $(V, \mathbb{C})$
(iv) Multiplicative identity (1 is the same for both $\mathbb{R}$ and $\mathbb{C}$)
(v) Distribution inherited.

Then φ_1 is an extension of ψ_1. Define

$$\varphi(x) = \varphi_1(x) - i\varphi_1(ix) \qquad \forall x \in V$$

Clearly φ is an extension of ψ on V since

$$\varphi(x) = \varphi_1(x) - i\varphi_1(ix) = \psi_1(x) - i\psi_1(ix)$$

Ex: φ is linear.

To show $\|\varphi\| \leq \|\psi\|$.

$$|\varphi(x)|^2 = \varphi(\overline{\varphi(x)}\, x) \xleftarrow{\text{real}} = \varphi_1(\overline{\varphi(x)}\, x) \leq \|\psi\| \, \|\overline{\varphi(x)}\, x\|$$

$$= \|\psi\| \|x\| \, |\cancel{\overline{\varphi(x)}}|$$

$$\Rightarrow \|\varphi\| \leq \|\psi\|.$$

In fact this implies $\|\varphi\| = \|\psi\|$. Why?

Prop: $\|v\|_V = \max \{ |\varphi(v)| : \varphi \in V' \quad \|\varphi\| = 1 \}$

Pf: Let $U = \{\alpha v : \alpha \in F\}$ and $\psi(\alpha v) = \alpha \|v\|$

ψ is linear. By Hahn-Banach ψ can be extended to φ with

$\|\varphi\| = \|\psi\| = 1$, and $\varphi(v) = \|v\|$ by definition.

For any other φ, $|\varphi(v)| \leq \|\varphi\| \|v\| = \|v\|$.

Prob : $\mathcal{U} \subset V$ subspace, $\| \ \|$ norm. Let $h \in V$. Then $h \in \overline{\mathcal{U}}$ iff $\varphi(h) = 0$ $\forall \varphi \in V'$, st $\varphi|_{\mathcal{U}} = 0$

Pf: If $h \in \overline{\mathcal{U}}$, $\varphi|_{\mathcal{U}} = 0$, $\varphi \in V'$ then $\varphi(h) = 0$ by continuity.

If $h \notin \overline{\mathcal{U}}$, let $\psi : \mathcal{U} + \mathbb{F}h \longrightarrow \mathbb{F}$ $\quad \psi(x + \alpha h) = \alpha$

Then $\psi(x) = 0$ $\forall x \in \mathcal{U}$. No the null $(\psi) = \mathcal{U}$.

Since $h \notin \overline{\mathcal{U}}$ $\overline{\text{null}(\psi)} \neq \mathcal{U} + \mathbb{F}h$. So ψ is a bounded fn on $\mathcal{U} + \mathbb{F}h$.

Hahn-Banach $\Rightarrow$ ψ can be extended to φ on V, $\varphi|_{\mathcal{U}} = 0$ but $\varphi(h) \neq 0$.